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ANISOTROPIC LAYERS WITH THROUGH-THICKNESS THERMAL AND MATERIAL VARIATIONS

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Abstract

A thermoelastic solution is derived for the problem of an inhomogeneous, fully anisotropic layer subject to an arbitrary non-uniform through-thickness temperature variation. The material properties vary smoothly through the thickness.

1. Introduction

The analysis of thermal stresses in anisotropic media has been of technological interest since the application of composites in the aerospace industry. In this context, multi-layered slabs or laminates are generally employed, the unidirectional distribution of the reinforcing fibers being varied from lamina to lamina. Motivated by the advantageous properties of these structures, an arsenal of techniques for the thermal stress analysis of anisotropic plates with discontinuous material properties has been developed [1, 2]. Central to these approaches is the representation of each composite lamina as an equivalent homogeneous material, according to one of the numerous available effective medium theories [3]. Analyzing the composite material in its effective representation yields the average, or macro-stresses. The determination of the local or micro-stress field for multi-phase media has received considerable attention as well [4 and references therein, 5, 6].

A common feature of the vast majority of micro- and macro-stress analyses is that the material inhomogeneities appear solely in a discontinuous fashion. By contrast, materials with properties that smoothly vary with position are attracting ever-increasing attention in the contemporary technology. This is the case for compositionally modulated composites, for which the distributions of the embedded phases are designed so that the macroscopic response is optimized for a prescribed application. A typical problem for this class of materials - known as Functionally Gradient Materials, or FGM - is the minimization of the thermal stresses generated during fabrication and subsequent service [7].

Problems involving layers of smoothly inhomogeneous materials are frequent in the areas of microelectronics, sensors, and of coating technology, where anisotropic thin films are commonly encountered not only in monocrystalline form, but also as multi-crystalline structures - like the micron-scale electromechanical devices, or MEMS [8] - and as larger, partially polycrystalline ceramic coatings. These typically present a continuous through-thickness variation in the microstructural portrait, which is a function of the deposition parameters [9].

With this background, in this paper the thermoelastic problem of an inhomogeneous anisotropic layer with material properties that vary smoothly through the thickness is examined. The problem is formulated in Section 2, and subsequently solved, for the case ofan arbitrary through-thickness temparature variation, via a semi-inverse technique, relying on the assumptions of simply-connectedness of the body. Some general properties of the solution are discussed in Section 4, where applications to specific classes of materials are also presented.

While this research was mainly motivated by the described developments in the fields of composite materials and thin films, where the analysis is performed on a point-wise homogenized representation of the materials [7,10], the present solution is applicable for the analysis of materials with chemical composition gradients and/or temperature-dependant material properties. Classical treatments for these media are found in [2,11,12].

2. Problem generalities.

Let the plate $-h \leq x_3 \leq h$, with lateral boundary $F(x_1,x_2) = 0$, be subject to a through-thickness thermal strain field

$$\underline{\epsilon}^* = \underline{\alpha}\,\delta T(x_3) \tag{1}$$

where $\underline{\alpha}$ and δT denote the thermal expansion tensor and the temperature variation fields, respectively. Let the constituent material be arbitrarily anisotropic and inhomogeneous, with material properties smoothly varying with position. The plate is not subjected to any external mechanical load: The body forces and the boundary tractions are zero everywhere. Introducing the plane stress assumption

$$\tau_{i3} = 0, \qquad i = 1,2,3 \tag{2}$$

together with the assumption of independence of the stress field on the in-plane coordinates, the equations of equilibrium are automatically satisfied, and the only field equations governing the problem are

$$e_{ijk} e_{lmn} \left(S_{jm\alpha\beta} \, \tau_{\alpha\beta} + \alpha_{jm} \, \delta T \right)_{,kn} = 0 \tag{3}$$

representing geometric compatibility for a simply connected body. Here and below, $\underline{e}$ is the permutation symbol, and $\underline{S}$ is the tensor of elastic compliances, which is related to the total strain $\underline{\epsilon}$, the thermal strain $\underline{\epsilon}^*$ and the stress $\underline{\tau}$ as

$$\underline{\epsilon} = \underline{S}\underline{\tau} + \underline{\epsilon}^* . \tag{4}$$

Following current use, we indicate partial differentiation of the function $f(\underline{x})$ with respect to x_n by $f_{,n}$, and employ the summation convention.

3. The case of material properties varying along x_3 only.

If the material properties $\underline{S}$ and $\underline{\alpha}$ are functions of x_3 only, then (3) simplifies to

$$e_{ij3} e_{km3} \left(S_{jm\alpha\beta} \, \tau_{\alpha\beta} + \alpha_{jm} \, \delta T \right)_{,33} = 0, \tag{5}$$

the only non-trivial independent components of which are

$$(S_{\alpha\beta\gamma\delta}\ \tau_{\gamma\delta} + \alpha_{\alpha\beta}\ \delta T)_{,33} = 0 \qquad (6)$$

for $(\alpha,\beta) = (1,1), (1,2), (2,2)$. Integration of these three equations yields

$$S_{\alpha\beta\gamma\delta}\ \tau_{\gamma\delta} = P_{\alpha\beta}\ x_3 + Q_{\alpha\beta} - \alpha_{\alpha\beta}\,\delta T \equiv A_{\alpha\beta} \qquad (7)$$

where $P_{\alpha\beta}$ and $Q_{\alpha\beta}$ are six unspecified constants. Equation (7) may be rewritten as

$$\underline{S}\ \underline{\tau} = \underline{A}, \qquad (8)$$

where $\underline{S}$ and $\underline{A}$ are defined as

$$\underline{S} \equiv \begin{bmatrix} S_{12} & 2S_{26} & S_{22} \\ S_{16} & 2S_{66} & S_{26} \\ S_{11} & 2S_{16} & S_{12} \end{bmatrix}, \qquad \underline{A} = \begin{bmatrix} A_2 \\ A_6 \\ A_1 \end{bmatrix} \qquad (9)$$

Here, the usual plane stress notation convention is adopted, identifying the pairs of tensorial indices 11, 12 and 22 with the single indices 1,6, and 2, respectively. This convention will be employed in the sequel, whenever advantageous.

Solving (8) yields the in-plane stresses

$$\tau_{\alpha\beta} = D_{\alpha\beta} / D \tag{10}$$

where $D \equiv \det \underline{S}$ (assumed non zero everywhere), and D_{11}, D_{12}, and D_{22} are the determinants of the matrices obtained from $\underline{S}$ upon substitution of the first, second and third column with the $\underline{A}$ vector, respectively.

The stress-free boundary conditions on $x_3 = \pm h$ are automatically satisfied, given (2), while the analogous conditions on the lateral boundary are violated, in general, since the stresses $\tau_{\alpha\beta}$ are non-vanishing functions of x_3 only. However, the constants $P_{\alpha\beta}$, $Q_{\alpha\beta}$ may be specified so that

$$\int_{-h}^{h} \tau_{\alpha\beta}\, dx_3 = \int_{-h}^{h} \tau_{\alpha\beta}\, x_3 dx_3 = 0 \qquad \text{for } \alpha,\beta = 1,2 \tag{11}$$

These six conditions ensure that the resultant force and moment acting on any line section of the lateral boundary vanish. Thus, a Saint-Venant-type solution is obtained. The equations (11) are equivalent to the linear system

$$\left[\begin{array}{c|c} \underline{R}^1 & \underline{R}^0 \\ \hline \underline{R}^2 & \underline{R}^1 \end{array}\right] \left[\begin{array}{c} \underline{P} \\ \hline \underline{Q} \end{array}\right] = \left[\begin{array}{c} \underline{F} \\ \hline \underline{G} \end{array}\right] \tag{12}$$

where the following vectors are introduced

$$
\underline{F} \equiv \begin{bmatrix} \int [\alpha_{22} Z_{11} + \alpha_{11} Z_{31} - \alpha_{12} Z_{21}]\, dx_3 \\ \int [\alpha_{22} Z_{12} + \alpha_{11} Z_{32} - \alpha_{12} Z_{12}]\, dx_3 \\ \int [\alpha_{22} Z_{13} + \alpha_{11} Z_{33} - \alpha_{12} Z_{23}]\, dx_3 \end{bmatrix}
$$

(13)

$$
\underline{G} \equiv \begin{bmatrix} \int [\alpha_{22} Z_{11} + \alpha_{11} Z_{31} - \alpha_{12} Z_{21}]\, x_3\, dx_3 \\ \int [\alpha_{22} Z_{12} + \alpha_{11} Z_{32} - \alpha_{12} Z_{12}]\, x_3\, dx_3 \\ \int [\alpha_{22} Z_{13} + \alpha_{11} Z_{33} - \alpha_{12} Z_{23}]\, x_3\, dx_3 \end{bmatrix}
$$

$$
\underline{P}^T \equiv (P_{22}, P_{12}, P_{11}) \quad ; \quad \underline{Q}^T \equiv (Q_{22}, Q_{12}, Q_{11}) \tag{14}
$$

together the 3x3 matrices $\underline{R}^n$, $n = 0,1,2$, defined by

$$
R^n_{\alpha\beta} \equiv \int x_3^n \; Z_{\beta\alpha} \; dx_3 \, . \tag{15}
$$

In equations (13,15), $Z_{\beta\alpha}$ denotes D times the determinant of the cofactor matrix associated with the element in row α and column β of the matrix $\underline{S}$, and all integrations are performed over the entire thickness of the layer.

Once the constants $P_{\alpha\beta}$, $Q_{\alpha\beta}$ are found by solving (11), the stress field is given by (10), and the strain field is obtained through (4). Integration of the strain-displacement relation yields the displacements:

$$
\begin{aligned}
u_1 &= x_1 \epsilon_{11} + x_2 \epsilon_{12} + 2 \int \epsilon_{13}\, dx_3 \\
u_2 &= x_2 \epsilon_{22} + x_1 \epsilon_{12} + 2 \int \epsilon_{23}\, dx_3 \\
u_3 &= \int \epsilon_{33}\, dx_3 - (x_1^2 \epsilon_{11,3} + x_2^2 \epsilon_{22,3} + 2\, x_1 x_2 \epsilon_{12,3})/2
\end{aligned} \tag{16}
$$

4. Applications and Discussion.

Some general features of the solution presented in Section 3 are now discussed, prior to the consideration of specific subcases:

a) Necessary conditions for the point-wise vanishing of the stresses are

$$(\alpha_{\alpha\beta}\ \delta T)_{,33} = 0 \qquad \alpha,\beta = 1,2, \qquad (17)$$

as may be deduced from (6). Thus, the material inhomogeneity may cause the stresses to be non-zero even for a uniform temperature variation.

b) In general, all strain components are non-zero, vary through the thickness only, and differ from the corresponding components of $\underline{\epsilon}^{*}$.

c) The curvatures $w_{\alpha\beta} \equiv u_{3,\alpha\beta}$ depend on x_3 only. The principal curvatures are generally different.

Explicit solutions are now given for some classes of materials.

(i) *Case 1:* $\underline{S}$ *is an even function of* x_3.

If all components of the tensor $\underline{S}$ exhibit an even functional dependence on x_3, then all $R^0_{\alpha\beta}$ and $R^2_{\alpha\beta}$ vanish, in the present coordinate system, and the $P_{\alpha\beta}$ decouple from the $Q_{\alpha\beta}$. It is found that

$$P_{11} = \{F_1(R^1_{31}\, R^1_{22} - R^1_{21}\, R^1_{32}) + F_2(R^1_{31}\, R^1_{12} - R^1_{11}\, R^1_{32}) + F_3(R^1_{21}\, R^1_{12} - R^1_{11}\, R^1_{22})\} \,/\, D^1$$

$$P_{22} = \{F_1(R^1_{32}\, R^1_{23} - R^1_{22}\, R^1_{33}) + F_2(R^1_{32}\, R^1_{13} - R^1_{12}\, R^1_{33}) + F_3(R^1_{22}\, R^1_{13} - R^1_{12}\, R^1_{23})\} \,/\, D^1 \qquad (18)$$

$$P_{12} = \{F_1(R^1_{31}\, R^1_{23} - R^1_{21}\, R^1_{33}) + F_2(R^1_{31}\, R^1_{13} - R^1_{11}\, R^1_{33}) + F_3(R^1_{21}\, R^1_{13} - R^1_{11}\, R^1_{23})\} \,/\, D^1$$

The constants $Q_{\alpha\beta}$ are obtained by substituting G_i for F_i in these expressions. In (18), D^1 is the determinant of $\underline{R}^1$.

(ii) *Case 2:* $\underline{S}$ *is an odd function of* x_3.

If all components of the tensor $\underline{S}$ exhibit an odd functional dependence on x_3, then $\underline{R}^1$ vanishes, and (12) decouples again. The $Q_{\alpha\beta}$ are now given by the right-hand-sides of (18), upon substituting $\underline{R}^1$ and D^1 with $\underline{R}^0$ and D^0, while the $P_{\alpha\beta}$ are again obtained from (18), but upon substituting F_i, $\underline{R}^1$ and D^1 with G_i, $\underline{R}^2$ and D^2.

(iii) *Case 3: Homogeneous material properties.*

While equations (18) and their analogues for the $Q_{\alpha\beta}$ also apply to problems involving homogenous materials, a different choice of constants leads, for this special case, to the following simpler formulation:

If the material properties $\underline{S}$ and $\underline{\alpha}$ are spatial constants, the only non-trivial independent conditions obtained from (3) are

$$S_{\alpha\beta\gamma\delta} \; \tau_{\gamma\delta,33} + \alpha_{\alpha\beta} \; \delta T_{,33} = 0 \tag{19}$$

for $(\alpha,\beta) = (1,1), (1,2), (2,2)$. The solution of these is

$$\tau_{\alpha\beta,33} = - \hat{D}_{\alpha\beta} \, \delta T_{,33} \, / \, D, \tag{20}$$

and thus

$$\tau_{\alpha\beta} = - \hat{D}_{\alpha\beta} \, \delta T / D + K'_{\alpha\beta} \, x_3 + K''_{\alpha\beta} \tag{21}$$

where the $\hat{D}_{\alpha\beta}$ are obtained from the $D_{\alpha\beta}$ upon substitution of $A_{\alpha\beta}$ with $\alpha_{\alpha\beta}$, and the $K'_{\alpha\beta}$, $K''_{\alpha\beta}$ are constants of integration. These constants are determined by imposing (11). Explicitly, this procedure yields

$$K'_{\alpha\beta} = + \, p \, \hat{D}_{\alpha\beta} \, M_T, \qquad K''_{\alpha\beta} = + \, q \, \hat{D}_{\alpha\beta} \, R_T \tag{22}$$

where

$$p \equiv 3 \, (2h^3 D)^{-1}, \qquad q \equiv (2hD)^{-1}, \tag{23}$$

and

$$R_T \equiv \int_{-h}^{h} \delta T(x_3) \; dx_3 \; , \qquad M_T \equiv \int_{-h}^{h} x_3 \; \delta T(x_3) \; dx_3 \tag{24}$$

For a linear thermal variation field the stresses are zero, in accordance with the exact thermoelastic theory [13], and in constrast with the inhomogeneous case. Actually, equations (22-24) may be obtained by imposing that the stresses be zero under arbitrary linear thermal loading. If this procedure is adopted, the Saint-Venant conditions (11) are then identically true. The displacements for the homogeneous case are again calculated by (16).

Specializing the material to be orthotropic in the natural reference frame,

$$D = 2 \; S_{66} \; (S_{12}^2 - S_{11} \; S_{22}),$$

$$\hat{D}_{11} = 2 \; S_{66} \; (\alpha_{11} \; S_{22} - \alpha_{22} \; S_{12}),$$

$$\hat{D}_{12} = 0 \tag{25}$$

$$\hat{D}_{22} = 2 \; S_{66} \; (\alpha_{22} \; S_{11} - \alpha_{11} \; S_{12}),$$

and thus the in-plane shear stress vanishes. If the material is further specialized to be transversely isotropic around x_3,

$$\hat{D}_{\alpha\beta} / D = \alpha\, \delta_{\alpha\beta}\, \dot{/}\, (S + S_{12}), \tag{26}$$

where α is the in-plane thermal expansion coefficient, $\delta_{\alpha\beta}$ is Kronecker's symbol, and $S \equiv S_{11} = S_{22}$. Clearly, $\tau_{11} = \tau_{22}$ in this case. If the material is isotropic, the solution of [14] is recovered.

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